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Abstract

Rubinstein (1989)'s paradox comes from the nature of games and players' ability of reasoning. We consider alternative assumptions to get his result. While we discard the assumptions that players know the probability of nature's move and the probability of failure of sending e-mail, bounded rationality of players and players' attitude to risks are incorporated in our model. We show that his result is obtained even though each player cannot operate the mathematical induction. Our model does not allow players to have richer information than Rubinstein's original model, but it is more natural one to describe our reasoning process.

1 Introduction

Common knowledge assumptions have been used in game-theoretical analyses explicitly or implicitly. Usually the following basic common knowledge is assumed. One is that the structure of a game is common knowledge. The other is that the fact that all players are Bayesian rational is common knowledge. In a view of decision-theoretical foundation of noncooperative games, these assumptions are essential. It, however, is hard to understand the significance of the assumptions because it is represented by infinite layers of knowledge like, 'I know that you know that I know ... and so on ad infinitum. Are there differences of outcomes of the games between under finite layers of knowledge (almost common knowledge) of the game and under common knowledge of the game? Rubinstein (1989) showed that assumptions of common knowledge and almost common knowledge of a game lead completely to different results, respectively.

In Rubinstein (1989) he showed that players cannot coordinate efficient strategies if the game is almost common knowledge, while it is possible when the game is common knowledge. He called the result "paradox" because it is counterintuitive when the game is sufficiently known each other (but it is not common knowledge). The discussion assumes implicitly that players can coordinate when the game is common knowledge. In this short paper we reconsider whether the paradox is really "paradox" by using non equilibrium model and shall show that the paradox comes from different level of difficulty to coordinate strategies and also players' ability to infer the opponents' knowledge.

As Rubinstein (1989) mentioned, the mathematical induction might not match to our intuitive reasoning. I guess that the disagreement occurs due to the different points of view of analysts and of players involved in the game. It is natural that our intuition is formed at realized state of the nature. Each player only realizes possible states at each state according to his information partition. For players to conduct the mathematical induction, they have to be third-party's position. Thus the fact makes us feel the discrepancy when we compare prescribed behavior of the game with our intuition. We do not use the mathematical induction to get outcomes of the game. By dropping the mathematical induction, we know what players would know and decide as their strategies at a realized state of the nature.

Now assume that in some game there are multiple equilibria, and players try to compare the equilibria. We use the equilibrium concept just only for players' reference point. At the beginning of comparing equilibria players should know what are equilibria. As Brandenburger and Dekel (1989) showed, it is not guaranteed for players to choose equilibrium strategies under basic common knowledge assumptions without players' internal consistent beliefs. Thus players would not know which equilibrium will occurs decision-theoretically under two basic common knowledge assumptions. I shall present that it is appropriate for players to consider possible rationalizable strategy combinations to compare equilibria. I shall measure the difficulty to coordinate strategies according to risk dominance, that is pair-wise comparison of equilibria.

Notice that the two basic common knowledge assumptions are used to derive rationalizable strategies. In Rubinstein's e-mail game common knowledge assumption of the game does not hold at each state of the nature. Then what are appropriate rationalizable strategies without assumption that the game is common knowledge? I shall construct solid decision problems under such a situation.

2 The Electronic Mail Game

Consider Rubinstein (1989)'s e-mail game. According to the nature's move, a state a is selected with probability $1-p$ and a state b with probability p, where $p<\frac{1}{2}$. At the state s, $s=\{a, b\}$, a game G_s is played by player 1 and player 2. The payoffs in the games are as follows.

state a: G_a

	A	B
A	M, M	1, −L
B	−L, 1	0, 0

state b: G_b

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

Only player 1 knows the true state. If state b occurs, 1 sends e-mail to 2 in order to report the true state. Each player replies to the opponent that he received the message. Due to a technical problem, the message fails to reach with probability $\epsilon > 0$. If state a occurs, 1 does not send the message.

Both games represent coordination problems. But G_a is different from G_b in the sense that there exists dominant strategies in G_a, but no such strategies exist in G_b. Thus no wonder that players in G_b cannot coordinate their strategies even when the game is common knowledge because of the difficulty of coordination. Then can we say that Rubinstein's result is really "paradox"? Let's consider the properties of each game, respectively.

Now assume that player 1 and player 2 will play game G_a, and that this fact is common knowledge. In addition (throughout this paper), we assume that all players are (Bayesian) rational, and the fact that all players are (Bayesian) rational is common knowledge. Under these assumptions we can consider players' rationalizable strategies.

Let $R_1^0 \equiv A_1 = \{A, B\}$, $R_2^0 \equiv A_2 = \{A, B\}$, and

$$B_i(\pi_{-i}) \equiv \mathrm{argmax}_{a_i \in A_i} U_i(a_i, \pi_{-i}),$$

where A_i is i's set of possible actions, U_i is player i's payoff function and π_{-i} is player i's belief over player $-i$'s actions. Then construct R_i^k, $k \in \mathrm{N}$ recursively as follows.

$$R_i^k \equiv B_i(\Delta(R_j^{k-1})),\ i \neq j,\ i, j \in \{1, 2\},$$

where $\Delta(X)$ is a set of probability distributions over a set X. Define

$$R_i \equiv \lim_{k\to\infty} R_i^k = \bigcap_{k=0}^{\infty} R_i^k.$$

Then

$$R_1 = R_2 = \{A\}.$$

Thus only action A is rationalizable strategy for both players in G_a.

Next instead assume that G_b is played, and that this fact is common knowledge. Then

$$R_1 = R_2 = \{A, B\}.$$

Both actions A and B are rationalizable strategies for both players under the assumption that game G_b is common knowledge. In contrast to game G_a, players cannot coordinate their strategies decision-theoretically in the game G_b.

Now introduce a communication mechanism via e-mail that forms players' knowledge. The set of states of the nature Ω, induced by the mechanism is

$$\Omega = \{(a, 0, 0) \text{and} (b, t, t') \mid t > 0 \text{ and } t' \text{ is either } t \text{ or } t-1\}.$$

Players' information partitions are represented by

$$\mathcal{P}_1 = \{\{(a, 0, 0)\}, \{(b, 1, 0), (b, 1, 1)\}, \{(b, 2, 1), (b, 2, 2)\}, \ldots\},$$

$$\mathcal{P}_2 = \{\{(a, 0, 0), (b, 1, 0)\}, \{(b, 1, 1), (b, 2, 1)\}, \{(b, 2, 2), (b, 3, 2)\}, \ldots\}.$$

Notice that both game G_a and G_b cannot become common knowledge.

When $T_1 = t$, 1 considers that $T_2 = t-1$ and $T_2 = t$ are possible from his information partition. Assume that 1 is trying to choose his action by using the mathematical induction that indicates that 2 will take some action as induction hypothesis. But the induction hypothesis is based on the fact that 2 concludes to choose the action according to also the induction hypothesis that 1 will choose some action at $T_1 = t-2$. Notice the state $T_1 = (t-2)$ (that is, $(b, t-2, t-3)$ or $(b, t-2, t-2)$) is impossible state from 1's point of view at $T_1 = t$. Certainly 1 knows that 2 considers $T_1 = t - 2$ possible. Does player 1 use the induction hypothesis that is based on 2's knowledge that comes from impossible state from 1's point of view?

We assume that people try to justify their actions and beliefs according to actually realized states from the players' point of view. It means that players do not use the mathematical induction to choose their actions.

3 Example

In this section we consider the e-mail game that is different from Rubinstein's in the sense that the probability of nature's move and the probability of failure of sending e-mails are not known to players. We shall describe players' conceivable rational strategies given their knowledge, then how they behave with mind of risk to take strategies.

Assume a state $\omega = (b, 2, 1)$ happened. Then player 1 knows G_b, player 2 knows G_b, 1 knows that 2 knows G_b, but 2 does not know that 1 knows that 2 knows G_b, and 1 does not know that 2 knows that 1 knows that 2 knows G_b. Under the state of the nature how do players choose their optimal strategies decision-theoretically given their knowledge about the game and their knowledge of their knowledge of the game and so on? Since 1 knows G_b and knows that 2 knows G_b, he can calculate up to $R_1^1 = \{A, B\}$ for sure as a set of his rational choices depends only on his knowledge. Also since 2 knows G_b, the set of his rational strategies is $R_2^0 = \{A, B\}$. Call these sets of strategies sets of players' limited rationalizable strategies. Let $\tilde{R}_1 \equiv R_1^1$ and $\tilde{R}_2 \equiv R_2^0$.

Now assume that players know the theory of games. Then each player knows which strategy combinations are equilibria, based on his limited rationalizable strategies. In addition assume that players try to coordinate their strategies using concept of risk dominance. Because $\tilde{R}_1=\{A, B\}$ and $\tilde{R}_2=\{A, B\}$, consider the following strategic game that is equivalent to the original game G_b.

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

Notice that it happens that the game is equivalent to the game G_b. In general, however, each player would consider a different game each other, depending on his knowledge of the game and his knowledge of the other player's knowledge of the game and so on. Each player might consider that the solution of the game must be coordinated equilibrium strategy combinations (A, A) or (B, B). This is justified by the fact that players want to coordinate their strategies from the structure of the game, and they know the theory of games.

By transforming the game to the game preserving the best-reply structure, we get the following strategic game (the unanimity game).

	A	B
A	L, L	0, 0
B	0, 0	M−1, M−1

Let π_i be the player $j\neq i$'s belief that player i takes action A. When 1 takes A, he gets $L\pi_2$ and when B, gets $(1-q_2)(M-1)$. Thus if $\pi_2<\frac{M-1}{L+M-1}$, B is better than A, and if $\pi_2>\frac{M-1}{L+M-1}$, A is better than B. Now $L>M$, so B is much more risky than A for player 1. Similarly if $\pi_1 > \frac{M-1}{L+M-1}$, then A is better than B for 2. If $\pi_1 < \frac{M-1}{L+M-1}$, then B is better than A for 2. Thus again B is much more risky than A for player 2. Both players have the same intensity to avoid risk, and they know this fact. Thus they would choose A. The consequence depends on players' beliefs of the other player's action. Here π_i is not determined so far. The determination will be discussed later.

As we expect $\frac{M-1}{M+L-1} \to 0$ as $L \to \infty$ so that no players think that they can coordinate their strategies.

4 The General Model

Given $\omega = (b, t, t-1)$, $t \geq 2$, 1 can calculate up to $R_1^{t-1} = B_1(\varDelta(R_2^{t-2}))$ because 1 does not know whether 2 receives 1's tth massage or not. On the other hand 2 can calculate up to $R_2^{t-2} = B_2(\varDelta(R_1^{t-3}))$ because 2 does not know whether 1 receives 2's $(t-1)$th message or not. Similarly at $\omega = (b, t, t)$, $t \geq 1$ 2 can calculate up to $R_2^{t-1} = B_2(\varDelta(R_1^{t-2}))$ for $t \geq 2$ and $R_2^0 = B_2(\varDelta(A_1))$ for $t=1$. Thus at $\omega = (b, t, t')$, $t' = t$ or $t-1$, $t \geq 1$, $t' \geq 1$, player 1 knows his limited rationalizable strategies in R_1^{t-1} and 2 knows his limited rationalizable strategies in $R_2^{t'-1}$. In order to induce R_1^{t-1}, 1 must knows R_2^{t-2} so that 1 thinks that 2's rationalizable strategies in R_2^{t-2}. Similarly to induce $R_2^{t'-1}$, 2 must know 1's rationalizable strategies in $R_1^{t'-2}$.

Assume that players know the theory of games so that they know which strategy combinations are Nash equilibria. By this assumption we do not mean that players can take Nash equilibrium strategies and moreover can coordinate their strategies because we do not assume players' internal consistent beliefs over other player's actions. We can only say that each player can recognize which are Nash equilibria.

Given $R_1^{t-1} \subseteq A_1$ and $R_2^{t-2} \subseteq A_2$, 1 knows Nash equilibrium $(a_1, a_2) \in R_1^{t-1} \times R_2^{t-2}$. Also given $R_2^{t'-1} \subseteq A_2$ and $R_1^{t'-2} \subseteq A_1$, 2 knows Nash equilibrium $(a_1, a_2) \in R_1^{t'-2} \times R_2^{t'-1}$. Notice that it is possible that players have considered different games. This is due to the fact that the game is G_b is not common knowledge.

Let

$$N_1 = \{(a_1, a_2) \in R_1^{t-1} \times R_2^{t-2} \mid a_i \text{ is the best response to } a_j, j \neq i, i, j \in \{1, 2\}\}.$$

That is, N_1 is a set of (pure strategy) Nash equilibrium from the point of view of 1 at $\omega = (b, t, t')$. Similarly let

$$N_2 = \{(a_1, a_2) \in R_1^{t'-2} \times R_2^{t'-1} \mid a_i \text{ is the best response to } a_j, j \neq i, i, j \in \{1, 2\}\}.$$

If $\sharp N_i \geq 2$, player i compares these equilibria by considering which equilibrium is likely to occur. That is, each player i is assumed to decide his optimal strategy in terms of risk dominance when there are multiple Nash equilibria from the point of view of i. According to Harsanyi and Selten (1988) let $\pi = (\pi_1, \pi_2)$ be subjective probability distribution over possible Nash equilibrium strategies, and $T_i(G_i, \pi)$ be the tracing procedure from the view of point of player i, where G_i is the game that player i thinks possible. Thus let

$$RD_1 = \{(a_1, a_2) \in R_1^{t-1} \times R_2^{t-2} \mid T_1(G_1, \pi) = (a_1, a_2), (a_1, a_2) \in R_1^{t-1} \times R_2^{t-2}\}.$$

$$RD_2 = \{(a_1, a_2) \in R_1^{t'-2} \times R_2^{t'-1} \mid T_2(G_2, \pi) = (a_1, a_2), (a_1, a_2) \in R_1^{t'-2} \times R_2^{t'-1}\}.$$

Notice that RD_i is not necessarily unique, nor are both RD_1 and RD_2 always the same.

Remark: The former represents that it is difficult to choose strategies in a coordination game. The latter indicates that as a result players cannot coordinate their strategies under the environment where players consider decision-theoretically another player's strategies.

Given $\omega = (a, 0, 0)$. Player 1 would take the dominant strategy A because he knows that the true state is a. Player 2 thinks that the true state is either $(a, 0, 0)$ or $(b, 1, 0)$. Thus $\tilde{R}_1 = \{A\}$, $\tilde{R}_2 = \{A, B\}$. 1 considers that 2's set of rational choices is A_2. 2 considers that 1's set of rational choices is A_1. Thus 1 considers the following game

	A	B
A	M, M	1, −L

2 considers

	A	B
A	M, M	1, −L
B	−L, 1	0, 0

or

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

Thus

$N_1 = \{(A, A)\}$.

$N_2 = \{(A, A)^a, (A, A)^b, (B, B)\}$.

where $(A, A)^s$ means the strategy combination (A, A) in the game G_s.

1 has no uncertainty in terms of risk. On the other hand, 2 tries to choose his strategy by risk comparison. Now $(A, A)^b$ risk dominates (B, B) in G_b. Consider the level of risks between games. In this case it seems to be trivial because 2 knows that 2 can play A safely in G_a and that 2 knows $(A, A)^b$ risk dominates (B, B). Thus 2 would play A because 2 does not care whether the game is G_a or G_b. Therefore both 1 and 2 choose strategy A.

Given $\omega=(b, 1, 0)$. Then $\tilde{R}_1=\{A, B\}$, $\tilde{R}_2=\{A, B\}$. 1 considers

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

2 considers

	A	B
A	M, M	1, −L
B	−L, 1	0, 0

or

	A	B
A	0, 0	1, −L
B	−L, 1	M, M

Also they choose A.

Proposition 1. *In the electronic mail game,*

(1) *when both players do not know the game that they will play, they can coordinate thier strategies at* $\omega=(a, 0, 0)$, *but not at* $\omega=(b, 1, 0)$.

(2) *As long as players know at least the game that they will play, they cannot coordinate in general, because the players' attitude to the risk to take an action deters them from taking efficient strategies whenever* $L > M$.

Proof. We have shown that the first part holds at $(a, 0, 0)$ and $(b, 1, 0)$. Notice $R_1 = \{A, B\}$ and $R_2 = \{A, B\}$ when the game G_b is common knowledge. For all $t \geq 1$, $t' \geq 1$, $\tilde{R}_1 = \{A, B\}$ and $\tilde{R}_2 = \{A, B\}$ at (b, t, t').

Thus

$$N_1 = \{(A, A), (B, B)\},$$

$$N_2 = \{(A, A), (B, B)\}.$$

$T_i(G_i, \pi) = (A, A)$ for $i = 1, 2$. Thus

$$RD_1 = \{(A, A)\},$$

$$RD_2 = \{(A, A)\}.$$

Therefore the players choose strategy A for any states of the nature.

The second part of Proposition 1 means that once the players realize the game G_b, they know for sure the difficulty of coordination in G_b even if they have sufficient knowledge of the game, so that the fact prevents them from coordinating their strategies.

5 Conclusion

In this paper we have reconsidered Rubinstein's paradox, and have showed that his result holds when each player can take actions that are feasible from the point of view of decision-theoretical foundation. Notice in Proposition 1 we did not tell how π is determined in detail because we know that the value of $T(G, \pi)$ is determined by considering the difference of Nash products in an unanimity game (Harsanyi and Selten (1988)).

If we could justify that π is uniformly distributed as in example, depending on players' beliefs of the opponent's actions, we could say that coordination is possible, but it would be difficult. We leave the problem open. The difficulty of the coordination comes from the fact that in the game G_b it is more difficult to coordinate than G_a. It is represented by that the concept of rationalizability does not reduce set of actions in the game G_b.

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