Performance Analysis of a Geom/G/1 Queueing System with General Limited Service and MAV

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Abstract

In this paper, a general limited service Geom/G/1 queueing system with multiple adaptive vacations (MAV) is analyzed. The Probability Generating Function (P.G.F.) of the queue length is obtained by using an embedded Markov chain with a regeneration cycle approach. The P.G.F. of the waiting time is also derived based on the independence between the arrival process and the waiting time. The probabilities for the system being in various states, such as "busy", "idle" and "vacation", are also derived. Finally, some special cases for the general limited service Geom/G/1 queueing system with multiple adaptive vacations are given to demonstrate the general properties of the queue model.

Keywords: Multiple adaptive vacations, general limited service, embedded Markov chain method, regeneration cycle approach, service cycle.

I. Introduction

Many Geom/G/1 queues with various vacation policies have been well investigated. Early research focused on exhaustive service policy where the server takes a vacation only if the system becomes idle. [1], [2]. However, multifarious non-exhaustive service policies also have important practical value in computer systems and communication networks [3], [4]. Several non-exhaustive service policies are introduced into the discussion of performance analysis of polling systems in [5], [6], where the server may take a vacation when there are customers waiting in the system. Stochastic decomposition results of vacation queues with general vacation policies are proved in [7], [8]. Many Geom/G/1 queues with a variety of vacation policies are systemically analyzed in [9]. Some research results of multi-server vacation queues are studied in [10].

In the adaptation of different application backgrounds, some new vacation policies were introduced into queues. A class of Geom/G/1 queues with exhaustive service and multiple adaptive vacations were studied in [10]. A discrete time Geom/G/1 queueing system with multiple adaptive vacations was investigated in [11]. The multiple adaptive vacation policy is a synthetic policy which generalizes several simple vacation policies. In this paper, we study a Geom/G/1 queueing system with non-exhaustive service and multiple adaptive vacations. Using an embedded Markov chain method and the regeneration cycle approach, we obtain the transformation formulae of the stationary queue length and the waiting time, and give stochastic decomposition structures of these stationary performance indices. General limited service Geom/G/1 queues with multiple vacations or single vacation in [9] are the special cases of our model.

II. Model description

Considering a classical Geom/G/1 queueing system, we introduce a general limited service and multiple adaptive vacation policy [10], [11]: a general limited service policy means that the number of customers which is served in every service period is not more than a determinate upper limit M, M is a positive integer. $Q_b^{(n)}$ is the number of customers in the system at the beginning instant of the nth service period, then the number of customers who will be served in the next service period is given by

$$\Phi = \min\{Q_b^{(n)},\ M\}.$$

The server will take H vacations with random length consecutively according to the assistant workload completed at that time. The number of the vacations is denoted by H, H is a positive integer random variable with the probability distribution h_j and the P.G.F. $H(z)$ as follows:

$$P(H=j)=h_j, \quad j\geq 1, \quad H(z)=\sum_{j=1}^{\infty}h_j z^j.$$

Vacation time length V_k ($k=1, 2, ..., H$) are independently identically distributed (i.i.d.) random variables. There are three cases as follows:

(1) If there are customers arriving during the kth vacation, $1\leq k\leq H$, the vacation period will stop in advance at the completion instant of the kth vacation. The server will begin a new service period, and the server will take

vacations again until Φ customers are served.

(2) If there are no customers arriving during H vacations, the server decides whether or not to enter an idle period based on the number of residual customers at the beginning instant of the service period after the Hth vacation finishes. If there are residual customers in the system, the server serves these residual customers immediately, then takes vacations.

(3) If there are no residual customers in the system after the Hth vacation finishes, the system enters an idle period. When there is a customer arrives, the server emerges from an idle state to serve the customer, then it begins to take vacations. The system will continually repeat the above process.

The basic assumptions of the model are given as follows:

(1) Suppose that customer arrivals can only occur at discrete time instant $t=n^-$, $n=0, 1, \ldots$. The beginning and ending of the service can only occur at discrete time instant $t=n^+$, $n=1, 2, \ldots$. The model is called a late arrival system. The inter-arrival time T is supposed to be an independently identically distributed (i.i.d.) discrete random variable following a geometric distribution with parameter $\lambda(0<\lambda<1)$. We can write the probability distribution of T as follows:

$$P(T=j)=\lambda\bar{\lambda}^{j-1}, \quad j=1,2,\ldots$$

where $\bar{\lambda}=1-\lambda$. We denote by C_n the number of customers arriving during the interval $[0,n]$, then C_n follows a Binomial distribution as follows:

$$P(C_n=j)=\binom{n}{j}\lambda^j\bar{\lambda}^{n-j}, \quad j=0,1,\ldots,n.$$

(2) The service time S of a customer is supposed to be an i.i.d. discrete random variable with a general distribution. The probability distribution s_j and the P.G.F. $S(z)$ of S are given as follows:

$$P(S_i=j)=s_j, \quad j\geq 1, \quad S(z)=\sum_{j=1}^{\infty}s_jz^j.$$

Let $E[S]$ and $E[S(S-1)]$ be the mean and the second factorial moment of S, then we have

$$\frac{1}{\mu}=E[S]=\sum_{i=0}^{\infty}is_i, \quad E[S(S-1)]=\left.\frac{d^2S(z)}{dz^2}\right|_{z=1}.$$

(3) The time length V of a vacation is a nonnegative i.i.d. discrete random variable with general probability distribution v_j and the P.G.F. $V(z)$ given by

$$P(V=j)=v_j, \quad j\geq 1, \quad V(z)=\sum_{j=1}^{\infty} v_j z^j.$$

where the mean $E[V]$ and the second factorial moment $E[V(V-1)]$ of V exist. $\rho=\frac{\lambda}{\mu}$ is the traffic intensity of the system.

Suppose that there is a single server in this system, and its capability is infinite. The interarrival time, the service time and the time length of a vacation are mutually independent. The service order is First Come First Served (FCFS). The model is denoted by Geom/G/1 (GL, MAV) queue, where GL and MAV represent the General Limited service and the Multiple Adaptive Vacations, respectively.

III. Performance analysis of the system

A. Preliminaries

According to non-exhaustive service policy, the transition probability matrix of the queue length $\{L_n, n\geq 1\}$ at the departure instant of a customer is different from the transition probability matrix of a classical Geom/G/1 queueing system at all states not just at boundary states alone, thus we can not simply apply the results of a Geom/G/1 boundary state variation model to study Geom/G/1 with the non-exhaustive service policy. The regeneration cycle approach is the most effective tool to apply to the stationary queue length of a system with a non-exhaustive service policy.

Let $L_v(t)$ represent the queue length process of a Geom/G/1 queue with a non-exhaustive service policy. The beginning instants of the service cycle are chosen as regeneration points when the number of customers is zero. The process $L_v(t)$ can be assumed to restart at these instants. If the system is positive recurrent, $L_v(t)$ will transit the zero state an infinite amount of times, therefore the system has infinite regeneration points. The interval between two adjacent regeneration points is defined as a regeneration cycle. A regeneration cycle may include several service cycles, and the length of a regeneration cycle is an i.i.d. random variable.

Lemma 3.1: If a stationary distribution exists, its P.G.F. can be written as

$$L_v(z)=\frac{E\left(\sum_{n=1}^{\Phi} z^{L_n}\right)}{E[\Phi]} \tag{1}$$

where L_n denotes the number of customers in the system at the nth departure instant in a service period.

The proof of the Lemma 3.1 is shown in [9], [10].

B. The number of customers at the beginning instant of the service cycle

It is easy to prove that $\{Q_b^{(n)},\ n\geq 1\}$ is a Markov chain, their transition probabilities are

$$P_{jk}=\begin{cases}\sum\limits_{r=k-j+M}^{\infty}\binom{r}{k-j+M}\lambda^{k-j+M}\bar{\lambda}^{r-k+j-M}P\left(B^{(M)}+V=r\right), & k\geq j-M>0\\ (1-H(V(\bar{\lambda})))\sum\limits_{r=k}^{\infty}\binom{r}{k}\lambda^{k}\bar{\lambda}^{r-k}P\left(B^{(j)}+V=r\right), & j\leq M,\ k\neq 1\\ (1-H(V(\bar{\lambda})))\sum\limits_{r=k}^{\infty}\binom{r}{k}\lambda^{k}\bar{\lambda}^{r-k}P\left(B^{(j)}+V=r\right)+H(V(\bar{\lambda})), & j\leq M,\ k=1\\ 0, & j>M,\ k<j-M\end{cases}\tag{2}$$

where $B^{(j)}+V$ is the sum of j service times and a vacation time, $0\leq j\leq M$. Define

$$q_k=\lim_{n\to\infty}P(Q_b^{(n)}=k),\qquad k\geq 0$$

where $\{q_k,\ k\geq 0\}$ is the distribution of the number of Q_b customers in the system at the beginning instant of the service period. From the equilibrium equation of the Markov chain, we have

$$\begin{aligned}q_0&=(1-H(V(\bar{\lambda})))\sum_{j=0}^{M}q_j\sum_{r=j}^{\infty}\binom{r}{0}\lambda^0\bar{\lambda}^rP\left(B^{(j)}+V=r\right)\\&=V(\bar{\lambda})(1-H(V(\bar{\lambda})))\sum_{j=0}^{M}q_j(S(\bar{\lambda}))^j,\end{aligned}\tag{3}$$

$$\begin{aligned}q_1&=\sum_{j=0}^{M}q_j\left((1-H(V(\bar{\lambda})))\sum_{r=j}^{\infty}\binom{r}{1}\lambda\bar{\lambda}^{r-1}P\left(B^{(j)}+V=r\right)+H(V(\bar{\lambda})\right)\\&\quad+q_{M+1}\sum_{r=j}^{\infty}\binom{r}{0}\lambda^0\bar{\lambda}^rP\left(B^{(M)}+V=r\right),\end{aligned}\tag{4}$$

$$\begin{aligned}q_k&=\sum_{j=0}^{M}q_j\left((1-H(V(\bar{\lambda})))\sum_{r=j}^{\infty}\binom{r}{k}\lambda^k\bar{\lambda}^{r-k}P\left(B^{(j)}+V=r\right)\right)\\&\quad+\sum_{j=M+1}^{k+M}\sum_{r=M}^{\infty}\binom{r}{k-j+M}\lambda^{k-j+M}\bar{\lambda}^{r-k+j-M}P\left(B^{(M)}+V=r\right),\quad k\geq 2.\end{aligned}\tag{5}$$

Defining the partial probability generating function

$$Q_M(z) = \sum_{k=0}^{M} q_k z^k.$$

Multiplying both sides of Eqs. (3), (4), (5) by z^0, z, z^k respectively, and taking the summation with respect to k, we obtain the P.G.F. of $\{q_k,\ k \geq 0\}$ as follows:

$$\begin{aligned} Q_b(z) &= (1 - H(V(\bar{\lambda})))Q_M(S(1-\lambda(1-z)))V(1-\lambda(1-z)) \\ &\quad + H(V(\bar{\lambda}))Q_M(1)z + \left(\frac{S(1-\lambda(1-z))}{z}\right)^M V(1-\lambda(1-z))(Q_b(z) - Q_M(z)). \quad (6) \end{aligned}$$

Arranging Eq. (6), we have that

$$\begin{aligned} Q_b(z) &= \frac{1}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\ &\quad \times \Big((1 - H(V(\bar{\lambda})))z^M Q_M(S(1-\lambda(1-z)))V(1-\lambda(1-z)) \\ &\quad - (S(1-\lambda(1-z)))^M Q_M(z) V(1-\lambda(1-z)) + H(V(\bar{\lambda}))Q_M(1)z^{M+1}\Big). \quad (7) \end{aligned}$$

To determine $Q_b(z)$, we need to obtain the coefficients q_0, q_1, ..., q_M of $Q_M(z)$ by using the Rouche theorem [12] and the Lagrange theorem [13]. In the denominator of Eq. (7), we define

$$f(z) = z^M, \quad g(z) = -V(1-\lambda(1-z))(S(1-\lambda(1-z)))^M.$$

For the probability distribution $\{c_k,\ k \geq 0\}$ of any non-negative integer random variable X and a sufficiently small $\varepsilon > 0$, in $|z| = 1+\varepsilon$, we have

$$|C(z)| = \left|\sum_{k=0}^{\infty} c_k z^k\right| \leq \sum_{k=0}^{\infty} c_k (1+\varepsilon)^k = \sum_{k=0}^{\infty} c_k (1 + k\varepsilon) + o(\varepsilon) = 1 + \varepsilon E[X] + o(\varepsilon).$$

Applying the above inequality to $g(z)$, we obtain

$$|g(z)| \leq 1 + (M\rho + \lambda E[V])\varepsilon + o(\varepsilon).$$

Obviously,

$$|f(z)| = (1+\varepsilon)^M = 1 + M\varepsilon + \mathrm{o}(\varepsilon).$$

If $\rho + \lambda E[V]M^{-1} < 1$, then $|f(z)| > |g(z)|$ in $|z| = 1+\varepsilon$. According to the Rouche theorem, $f(z)$ and $f(z)+g(z)$ have the same number of roots in $|z| = 1+\varepsilon$. Therefore, the denominator of Eq. (7) has M roots in $|z| = 1+\varepsilon$, where one root is $z = 1$, the other $M-1$ roots are given by applying the Lagrange theorem

$$z_m = \sum_{n=1}^{\infty} \frac{e^{\frac{2\pi mn}{M}i}}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(V(1-\lambda(1-z))(S(1-\lambda(1-z)))^M\right)^{\frac{n}{M}}\Bigg|_{z=0},$$
$$m = 1, 2, \ldots, M-1. \quad (8)$$

Because $Q_b(z)$ is analytic in $|z|<1$, the numerator of Eq. (7) has the same roots. Hence, $\{q_k,\ k=0, 1, \ldots, M\}$ satisfies the set of equations comprised of the following $M-1$ linear equations as follows:

$$\sum_{k=0}^{M} q_k \Big((1-H(V(\bar{\lambda})))z_m^M (S(1-\lambda(1-z_m)))^k V(1-\lambda(1-z_m))$$
$$-(S(1-\lambda(1-z_m)))^M z_m^k V(1-\lambda(1-z_m)) + H(V(\bar{\lambda}))z_m^{M+1}\Big) = 0,$$
$$m = 1, 2, \ldots, M-1. \quad (9)$$

Based on the normalization condition $Q_b(1)=1$ and by applying the L'Hospital rule in Eq. (7), we have

$$1 = \frac{(M(1-\rho)+H(V(\bar{\lambda}))(1-\lambda E[V]))Q_M(1) - ((1-\rho)+\rho H(V(\bar{\lambda})))Q_M'(1)}{M(1-\rho)-\lambda E[V]}. \quad (10)$$

From Eq. (10), we obtain the relation between $Q_M'(1)$ and $Q_M(1)$ as follows:

$$Q_M'(1) = \frac{\lambda E[V]-M(1-\rho)}{(1-\rho)+\rho H(V(\bar{\lambda}))} + \frac{M(1-\rho)+H(V(\bar{\lambda}))(1-\lambda E[V])}{(1-\rho)+\rho H(V(\bar{\lambda}))} Q_M(1). \quad (11)$$

According to Eq. (11), we obtain the Mth equation about q_k, $k=0, 1, 2, \ldots, M$ as follows:

$$\sum_{k=0}^{M} q_k \Big(M(1-\rho)+H(V(\bar{\lambda}))(1-\lambda E[V]) + k((1-\rho)+\rho H(V(\bar{\lambda})))\Big)$$
$$= M(1-\rho)-\lambda E[V]. \quad (12)$$

From Eqs. (3), (9) and (12), we obtain set of equations comprised by $M+1$ linear equations, thus we can resolve q_k, $k=0, 1, 2, \ldots, M$ and $Q_M(z)$. Taking derivatives for both sides of Eq. (7) with respect to z and by applying the L'Hospital rule, the mean number of customers at the beginning instant of the service cycle is given by

$$E[Q_b] = \frac{1}{2(M(1-\rho)-\lambda E[V])}$$
$$\times\Big\{\lambda^2 M E[S(S-1)] - \big(M(M-1)(1-\rho^2) + 2\lambda\rho M E[V]$$
$$+\lambda^2 E[V(V-1)]\big) + Q_M''(1)((1-H(V(\bar{\lambda})))\rho^2 - 1)$$
$$+Q_M'(1)\big((1-H(V(\bar{\lambda})))\big(\lambda^2 E[S(S-1)] + 2\lambda\rho E[V]\big)$$

$$-2\rho MH(V(\bar{\lambda}))M - 2\lambda E[V]\big) + Q_M(1)\big((1 - H(V(\bar{\lambda})))\big(M(M-1)$$
$$+2\lambda ME[V] + \lambda^2 E[V(V-1)]\big)\big)$$
$$-M(M-1)\rho^2 - \lambda^2 ME[S(S-1)] - 2\lambda\rho ME[V]$$
$$-\lambda^2 E[V(V-1)] + M(M+1)H(V(\bar{\lambda}))\big)\Big\}. \tag{13}$$

Combining $\Phi = \min\{Q_b, M\}$ and Eq. (11), we obtain that

$$E[\Phi] = \sum_{k=1}^{M} kq_k + M\sum_{k=M+1}^{\infty} q_k = Q'_M(1) + M(1 - Q_M(1))$$
$$= \frac{\lambda E[V] + \rho MH(V(\bar{\lambda})) + H(V(\bar{\lambda}))(1 - \rho M - \lambda E[V])Q_M(1)}{1 - \rho(1 - H(V(\bar{\lambda})))}. \tag{14}$$

The equilibrium condition of the system requires that the mean number of customers arriving in a service cycle is less than M, that means

$$\lambda(E[\Phi]E[S] + (1 - H(V(\bar{\lambda})))E[V]) + H(V(\bar{\lambda}))$$
$$= \rho E[\Phi] + \lambda E[V] + H(V(\bar{\lambda}))(1 - \lambda E[V]) < \mathrm{M}. \tag{15}$$

Therefore, the equilibrium condition of the system is given by

$$M - \rho E[\Phi] - \lambda E[V] - H(V(\bar{\lambda}))(1 - \lambda E[V]) > 0. \tag{16}$$

Based on the regeneration cycle approach and the expression of $Q_b(z)$, we obtain the stochastic decomposition structure of stationary performance measures for a general limited service Geom/G/1 queueing system with multiple adaptive vacations.

C. The stationary queue length and waiting time

Theorem 3.2: If $M - \rho E[\Phi] - \lambda E[V] - H(V(\bar{\lambda}))(1 - \lambda E[V]) > 0$ and $\rho + \lambda E[V]M^{-1} < 1$, the stationary queue length L_v in Geom/G/1 (GL, MAV) queueing system can be decomposed into three independent random variables:

$$L_v = L + L_d + L_r$$

where L is the stationary queue length in a classical Geom/G/1 queueing system [9], [10], its P.G.F. is

$$L(z) = \frac{(1-\rho)(1-z)S(1-\lambda(1-z))}{S(1-\lambda(1-z)) - z}. \tag{17}$$

The additional queue length L_d is the additional queue length of a Geom/G/1 queueing system with multiple adaptive vacations. The additional queue length L_r is the additional queue length resulting from the general limited service policy. P.G.Fs. $L_d(z)$ and $L_r(z)$ of the additional queue lengths L_d and L_r are given by

$$L_d(z) = \frac{1 - zH(V(\bar{\lambda})) - \dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(1-\lambda(1-z)) - V(\bar{\lambda}))}{\left(H(V(\bar{\lambda})) + \dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z)}, \tag{18}$$

$$\begin{aligned} L_r(z) = {} & \frac{\beta}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\ & \times \frac{1}{1 - V(1-\lambda(1-z)) + H(V(\bar{\lambda}))\big(V(1-\lambda(1-z)) - (1-V(\bar{\lambda}))z - V(\bar{\lambda})\big)} \\ & \times \Big(Q_M(S(1-\lambda(1-z)))\big(z^M(1 - V(1-\lambda(1-z))) \\ & + H(V(\bar{\lambda}))V(1-\lambda(1-z))(z^M - (S(1-\lambda(1-z)))^M)\big) \\ & - Q_M(z)(S(1-\lambda(1-z)))^M(1 - V(1-\lambda(1-z))) \\ & + H(V(\bar{\lambda}))Q_M(1)z\big((S(1-\lambda(1-z)))^M - z^M\big)\Big). \end{aligned} \tag{19}$$

proof: Because L_n is the number of customers at the departure instant of the nth customer in a service cycle, and A_k represents the number of customers arriving in the kth service period, then we have

$$L_n = Q_b - n + \sum_{k=1}^{n} A_k, \qquad n = 1, 2, \ldots, \Phi.$$

According to the definition of Φ, we have

$$\begin{aligned} E\left(\sum_{n=1}^{\Phi} z^{L_n}\right) & = E\left(\sum_{n=1}^{\Phi} z^{Q_b - n}(S(1-\lambda(1-z)))^n\right) \\ & = \sum_{k=1}^{M} P(Q_b = k)\sum_{n=1}^{k} z^{k-n}(S(1-\lambda(1-z)))^n \\ & \quad + \sum_{k=M+1}^{\infty} P(Q_b = k)\sum_{n=1}^{M} z^{k-n}(S(1-\lambda(1-z)))^n \\ & = \frac{S(1-\lambda(1-z))}{S(1-\lambda(1-z)) - z}\,\frac{1}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\ & \quad \times \Big(Q_M(S(1-\lambda(1-z)))\big(z^M(1 - V(1-\lambda(1-z))) \\ & \quad + H(V(\bar{\lambda}))V(1-\lambda(1-z))(z^M - (S(1-\lambda(1-z)))^M)\big) \\ & \quad - Q_M(z)(S(1-\lambda(1-z)))^M(1 - V(1-\lambda(1-z))) \\ & \quad + H(V(\bar{\lambda}))Q_M(1)z\big((S(1-\lambda(1-z)))^M - z^M\big)\Big). \end{aligned} \tag{20}$$

The empty summation is defined to be zero. Substituting Eqs. (14) and (20) into Eq. (1), we obtain the following equation:

$$
\begin{aligned}
L_v(z) = \frac{E[z^{L_n}]}{E[\Phi]} &= \frac{(1-\rho)(1-z)S(1-\lambda(1-z))}{S(1-\lambda(1-z))-z} \\
&\times \frac{1-zH(V(\bar{\lambda}))-\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(1-\lambda(1-z))-V(\bar{\lambda})}{\left(H(V(\bar{\lambda}))+\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z)} \\
&\times \frac{\beta}{z^M-(S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \\
&\times \frac{1}{1-V(1-\lambda(1-z))+H(V(\bar{\lambda}))\big(V(1-\lambda(1-z))-(1-V(\bar{\lambda}))z-V(\bar{\lambda}}
\\
&\times\Big(Q_M(S(1-\lambda(1-z)))\big(z^M(1-V(1-\lambda(1-z)))\\
&+H(V(\bar{\lambda}))V(1-\lambda(1-z))(z^M-(S(1-\lambda(1-z)))^M)\big)\\
&-Q_M(z)(S(1-\lambda(1-z)))^M(1-V(1-\lambda(1-z)))\\
&+H(V(\bar{\lambda}))Q_M(1)z\big((S(1-\lambda(1-z)))^M-z^M\big)\Big)\\
&= L(z)L_d(z)L_r(z). \qquad (21)
\end{aligned}
$$

where $\beta = \dfrac{\big(1-\rho(1-H(V(\bar{\lambda})))\big)\big[\lambda E[V]+H(V(\bar{\lambda}))(1-V(\bar{\lambda})-\lambda E[V])\big]}{(1-\rho)\big[\lambda E[V]+\rho MH(V(\bar{\lambda}))+H(V(\bar{\lambda}))Q_M(1)(1-\rho M-\lambda E[V])\big]}$, therefore we obtain the P.G.Fs. of the additional queue length. ■

Taking a derivative of $L_r(z)$ with respect to z, then applying the L'Hospital rule repeatedly and letting $z=1$, we obtain the mean additional queue length $E[L_r]$ as follows:

$$E[L_r] = \frac{\beta(ca-bd)}{2a^2}$$

where

$$
\begin{aligned}
a &= 2(M(1-\rho)-\lambda E[V])\times\big(H(V(\bar{\lambda}))(\lambda E[V]+V(\bar{\lambda})-1)-\lambda E[V]\big),\\
b &= 2(1-\rho)Q'_M(1)\big(\lambda E[V]+\rho MH(V(\bar{\lambda}))\big)\\
&\quad +2MQ_M(1)\big(\lambda E[V](1-\rho)(1+H(V(\bar{\lambda})))+\rho H(V(\bar{\lambda}))\big),\\
c &= 3(1-\rho)Q''_M(1)(\lambda E[V](1+\rho)+MH(V(\bar{\lambda})))\\
&\quad +3Q'_M(1)\big(\lambda^2 E[S(S-1)](MH(V(\bar{\lambda}))(1-\rho)+\lambda E[V])\\
&\quad -2\lambda\rho MH(V(\bar{\lambda}))E[V](1-\rho)+\lambda^2 E[V(V-1)](1-\rho)
\end{aligned}
$$

$$+\rho MH(V(\bar{\lambda}))((M-1)(1-\rho)-\lambda^2E[S(S-1)]\big)$$
$$+Q_M(1)\big(3E[V]M(M-1)(1-\rho^2)(\lambda+\rho)$$
$$+H(V(\bar{\lambda}))\big(M(M-1)(M-3)+\lambda^3ME[S(S-1)(S-2)]$$
$$-3\rho M(M-1)+3\lambda^3ME[S(S-1)]\big)\big),$$
$$d=3\big(M(M-1)(1-\rho^2)-\lambda^2ME[S(S-1)]-2\lambda\rho ME[V]$$
$$-\lambda^2E[V(V-1)]\big)\times\big(H(V(\bar{\lambda}))(\lambda E[V]+V(\bar{\lambda})-1)-\lambda E[V]\big)$$
$$+3\lambda^2E[V(V-1)](M(1-\rho)-\lambda E[V])(H(V(\bar{\lambda}))-1).$$

With Eq. (21) and applying the L'Hospital rule, we obtain the mean queue length $E[L_v]$ as follows:

$$E[L_v]=\rho+\frac{\lambda^2E[S(S-1)]}{2(1-\rho)}+\frac{\frac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda^2E[V(V-1)]}{2\left(H(V(\bar{\lambda}))+\frac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)}+\frac{\beta(ca-bd)}{2a^2}. \quad (22)$$

Based on the stochastic decomposition result of the queue length L_v, and the classical relation that the number of customers in the system at the departure instant of a customer is equal to the number of customers arriving in the sojourn time, we prove the stochastic decomposition result of the waiting time.

Theorem 3.3: If $M-\rho E[\Phi]-\lambda E[V]-H(V(\bar{\lambda}))(1-\lambda E[V])>0$ and $\rho+\lambda E[V]M^{-1}<1$, the stationary waiting time W_v in Geom/G/1 (GL, MAV) queueing system can be decomposed into three independent random variables:

$$W_v=W+W_d+W_r$$

where W is the stationary waiting time in a classical Geom/G/1 queueing system [9], [10], its P.G.F. is

$$W(z)=\frac{(1-\rho)(1-z)}{(1-z)-\lambda(1-S(z))}. \quad (23)$$

The additional delay W_d is the additional delay of a Geom/G/1 queueing system with multiple adaptive vacations. The additional delay L_r is the additional delay resulting from the general limited service policy. P.G.Fs. $W_d(z)$ and $W_r(z)$ of the additional delays W_d and W_r are given by

$$W_d(z)=\frac{\lambda-H(V(\bar{\lambda}))(z-\bar{\lambda})-\lambda\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(z)-V(\bar{\lambda}))}{\left(H(V(\bar{\lambda}))+\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z)}, \tag{24}$$

$$\begin{aligned}W_r(z)=&\frac{\beta}{(z-\bar{\lambda})^M-(\lambda S(z))^M V(z)}\\&\times\frac{1}{\lambda+H(V(\bar{\lambda}))(\bar{\lambda}-V(\bar{\lambda}))-\lambda V(z)(1-H(V(\bar{\lambda})))+H(V(\bar{\lambda}))(1-V(\bar{\lambda}))z}\\&\times\Big(zQ_M(S(z))\big((z-\bar{\lambda})^M(1-V(z))+H(V(\bar{\lambda}))V(z)((z-\bar{\lambda})^M\\&-(\lambda S(z))^M)\big)-\lambda^{M+1}Q_M\left(\frac{z-\bar{\lambda}}{\lambda}\right)S(z))^M(1-V(z))\\&+H(V(\bar{\lambda}))Q_M(1)(z-\bar{\lambda})((\lambda S(z))^M-(z-\bar{\lambda})^M)\Big).\end{aligned} \tag{25}$$

proof: Because the waiting time is independent of the input process after the arrival instant, the number of residual customers in the system after the departure instant is equal to the sum of the number of customers arriving in the waiting time W_v and the service time S for a Geom/G/1 (GL, MAV) model. Because of the independent increment property of the input process which follows a binomial distribution, the number of customers arriving in the waiting time and the service time are mutually independent. We then have

$$L_v(z)=W_v(1-\lambda(1-z))S(1-\lambda(1-z)). \tag{26}$$

Substituting $L_v(z)$ in Theorem 3.2 into Eq. (26), and letting $z'=1-\lambda(1-z)$, we obtain that

$$\begin{aligned}W_v(z')=&\frac{(1-\rho)(1-z')}{(1-z')-\lambda(1-S(z'))}\\&\times\frac{\lambda-H(V(\bar{\lambda}))(z'-\bar{\lambda})-\lambda\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}(V(z')-V(\bar{\lambda}))}{\left(H(V(\bar{\lambda}))+\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)(1-z')}\\&\times\Bigg\{\frac{\beta}{(z'-\bar{\lambda})^M-(\lambda S(z'))^M V(z')}\\&\times\frac{1}{\lambda+H(V(\bar{\lambda}))(\bar{\lambda}-V(\bar{\lambda}))-\lambda V(z')(1-H(V(\bar{\lambda})))+H(V(\bar{\lambda}))(1-V(\bar{\lambda}))z'}\\&\times\Big(z'Q_M(S(z'))\big((z'-\bar{\lambda})^M(1-V(z'))\\&+H(V(\bar{\lambda}))V(z')((z'-\bar{\lambda})^M-(\lambda S(z'))^M)\big)\end{aligned}$$

$$-\lambda^{M+1}Q_M\left(\frac{z'-\bar{\lambda}}{\lambda}\right)(S(z'))^M(1-V(z'))$$

$$\left.+H(V(\bar{\lambda}))Q_M(1)(z'-\bar{\lambda})((\lambda S(z'))^M-(z'-\bar{\lambda})^M)\right)\Bigg\}$$

$$=W(z')W_d(z')W_r(z'). \quad (27)$$

Displacing z' with z in the last equation in Eq. (24), the P.G.Fs. of any additional delay are Eqs. (24) and (25), respectively. ■

Taking a derivative of Eqs. (24) and (25) with respect to z, and using the L'Hospital rule, we obtain the mean waiting time as follows:

$$E[W_v]=\frac{\lambda E[S(S-1)]}{2(1-\rho)}+\frac{\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V(V-1)]}{2\left(H(V(\bar{\lambda}))+\dfrac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}\lambda E[V]\right)}+\frac{\beta(ca-bd)}{2\lambda a^2}. \quad (28)$$

D. The analysis of the service cycle

According to the definition of the number J of consecutive vacations [10], [11], we have that

$$P(J\geq 1)=1,$$
$$P(J\geq j)=P(H\geq j)P(V_1+V_2+\cdots+V_{j-1}\geq T)=(\mathrm{V}(\bar{\lambda}))^{j-1}\sum_{k=j}^{\infty}h_k,\quad j\geq 2. \quad (29)$$

Thus the P.G.F. $J(z)$ of J can be obtained as follows:

$$J(z)=1-\frac{1-z}{1-V(\bar{\lambda})z}(1-H(V(\bar{\lambda})z)). \quad (30)$$

The P.G.F. $V_G(z)$ of V_G which is the whole time length for every two consecutive vacations is given as follows:

$$V_G(z)=J(V(z))=1-\frac{1-V(z)}{1-V(\bar{\lambda})V(z)}(1-H(V(\bar{\lambda})V(z))). \quad (31)$$

Thus the mean vacation time $E[V_G]$ is given as follows:

$$E[V_G]=\frac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})}E[V]. \quad (32)$$

In a general limited service Geom/G/1 queueing system with multiple adaptive vacations, the server may stay in an idle period. The idle period is equal to zero at the completion instant of J vacations when one of the following three cases holds, i.e. (i) If there are customers arriving in the service period. (ii) If there are no customers arriving in the service period but there are customers arriving during the vacation period. (iii) If there are no customers arriving in the service period and the vacation period but there are residual customers that have resulted from a general limited policy. The idle period is equal to an inter-arrival time (because the inter-arrival time follows a geometric distribution with a memory-less property) if there are no customers arriving at the completion instant of J vacations and there are no residual customers resulting from a general limited policy. Let I_v be the time length of the idle period, the mean idle period $E[I_v]$ is given by

$$\begin{aligned} E[I_v] &= \frac{1}{\lambda} H(V(\bar{\lambda}))L_v(0) \\ &= \frac{(1-\rho)\beta H(V(\bar{\lambda}))\big(H(V(\bar{\lambda}))V(\bar{\lambda})Q_M(S(\bar{\lambda})) + Q_M(0)(1-V(\bar{\lambda}))\big)}{\lambda V(\bar{\lambda})\big(H(V(\bar{\lambda}))(1-V(\bar{\lambda})) + \lambda E[V](1-H(V(\bar{\lambda})))\big)}. \end{aligned} \tag{33}$$

The mean busy period is given by

$$\begin{aligned} E[S_\lambda] &= E[\Phi]E[S] \\ &= \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1-\rho M - \lambda E[V])}{\mu(1-\rho(1-H(V(\bar{\lambda}))))}. \end{aligned} \tag{34}$$

Let C be the interval between the beginning instant of two service periods, called a "service cycle". Then the mean service cycle $E[C]$ is given by

$$\begin{aligned} E[C] &= E[S_\lambda] + E[V_G] + E[I_v] \\ &= \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1-\rho M - \lambda E[V])}{\mu(1-\rho(1-H(V(\bar{\lambda}))))} \\ &\quad + \frac{(1-\rho)\beta H(V(\bar{\lambda}))\big(H(V(\bar{\lambda}))V(\bar{\lambda})Q_M(S(\bar{\lambda})) + Q_M(0)(1-V(\bar{\lambda}))\big)}{\lambda V(\bar{\lambda})\big(H(V(\bar{\lambda}))(1-V(\bar{\lambda})) + \lambda E[V](1-H(V(\bar{\lambda})))\big)}. \\ &\quad + \frac{1-H(V(\bar{\lambda}))}{1-V(\bar{\lambda})} E[V]. \end{aligned} \tag{35}$$

Let p_B, p_V and p_I be the probabilities of the server being at the states of busy, vacation and idle, respectively. We can give that

$$p_B = \frac{\lambda E[V] + \rho M H(V(\bar{\lambda})) + H(V(\bar{\lambda}))Q_M(1)(1-\rho M - \lambda E[V])}{\mu(1-\rho(1-H(V(\bar{\lambda}))))E[C]},$$
$$p_V = \frac{E[V](1-H(V(\bar{\lambda})))}{E[C](1-V(\bar{\lambda}))},$$
$$p_I = \frac{(1-\rho)\beta H(V(\bar{\lambda}))\big(H(V(\bar{\lambda}))V(\bar{\lambda})Q_M(S(\bar{\lambda})) + Q_M(0)(1-V(\bar{\lambda}))\big)}{\lambda V(\bar{\lambda})E[C]\big[H(V(\bar{\lambda}))(1-V(\bar{\lambda})) + \lambda E[V](1-H(V(\bar{\lambda})))\big]}. \tag{36}$$

IV. Special cases

In our model, if the random variable H follows different distributions, we can obtain some vacation queues with general limited service as special cases of the model presented in this paper as follows.

Example 4.1: A general limited service Geom/G/1 queueing system with multiple vacations.

If $H \to \infty$, our model turns into a general limited service Geom/G/1 queueing system with multiple vacations. There is not the idle state in the system, where $H(z)=0$ and $\beta=1$, therefore the P.G.Fs. of the additional queue length L_d and L_r, and the P.G.Fs. of the additional delay W_d and W_r correspond with the results given in [9], [10].

Example 4.2: A general limited service Geom/G/1 queueing system with a single vacation.

If $H=1$, our model turns into a general limited service Geom/G/1 queueing system with a single vacation, there is an idle state in the system, where $H(z)=z$ and β is equal to β_1 as follows:

$$\beta_1 = \frac{(1-\rho+\rho V(\bar{\lambda}))\big(\lambda E[V] + V(\bar{\lambda})(1-V(\bar{\lambda}) - \lambda E[V])\big)}{(1-\rho)\big(\lambda E[V] + \rho M V(\bar{\lambda}) + V(\bar{\lambda})Q_M(1)(1-\lambda E[V] - \rho M)\big)}.$$

Therefore, P.G.Fs. of the additional queue length L_d and L_r are given as follows:

$$L_d(z) = \frac{1 - V(1-\lambda(1-z)) + V(\bar{\lambda})(1-z)}{\big(V(\bar{\lambda}) + \lambda E[V]\big)(1-z)},$$

$$L_r(z) = \frac{\beta_1}{z^M - (S(1-\lambda(1-z)))^M V(1-\lambda(1-z))} \times \frac{1}{1-(1-V(\bar{\lambda}))(V(1-\lambda(1-z)) + V(\bar{\lambda})z) - (V(\bar{\lambda}))^2}$$

$$\times\Big(Q_M(S(1-\lambda(1-z)))\Big(z^M(1-V(1-\lambda(1-z)))+V(\bar{\lambda})$$
$$\times(z^M-(S(1-\lambda(1-z)))^M)\Big)$$
$$-Q_M(z)(S(1-\lambda(1-z)))^M(1-V(1-\lambda(1-z)))$$
$$+V(\bar{\lambda})Q_M(1)z\Big((S(1-\lambda(1-z)))^M-z^M\Big)\Big). \tag{37}$$

The P.G.Fs. of the additional delay W_d and W_r are given as follows:

$$W_d(z)=\frac{\lambda(1-V(1-\lambda(1-z)))+V(\bar{\lambda})(1-z)}{\big(V(\bar{\lambda})+\lambda E[V]\big)(1-z)},$$

$$W_r(z)=\frac{\beta_1}{\Big((z-\bar{\lambda})^M-(\lambda S(z))^M V(z)\Big)\big(\lambda(1-V(\bar{\lambda}))(1-V(z))+V(\bar{\lambda})(1-V(\bar{\lambda}))(1+z)\big)}$$
$$\times\Big(zQ_M(S(z))\Big((z-\bar{\lambda})^M(1-V(z))+V(\bar{\lambda})V(z)((z-\bar{\lambda})^M-(\lambda S(z))^M)\Big)$$
$$-\lambda^{M+1}Q_M\left(\frac{z-\bar{\lambda}}{\lambda}\right)(S(z))^M(1-V(z))$$
$$+V(\bar{\lambda})Q_M(1)(z-\bar{\lambda})\Big((\lambda S(z))^M-(z-\bar{\lambda})^M\Big)\Big). \tag{38}$$

Example 4.3: A general limited service Geom/G/1 (GL, MAV) queueing system with H follows a geometric distribution.

If the number of vacations follows a geometric distribution with parameters q, and $P_G(H=i)=q\bar{q}^{i-1}$, $i=1,2,\dots$, $q>0$, $\bar{q}>0$, $q+\bar{q}=1$, then $H(z)=\dfrac{qz}{1-\bar{q}z}$, where $H(V(\bar{\lambda}))=\dfrac{qV(\bar{\lambda})}{1-\bar{q}V(\bar{\lambda})}$, and β is a constant β_2 as follows:

$$\beta_2=\frac{\left(1-\rho\left(1-\dfrac{qV(\bar{\lambda})}{1-\bar{q}V(\bar{\lambda})}\right)\right)\left(\lambda E[V]+\dfrac{qV(\bar{\lambda})}{1-\bar{q}V(\bar{\lambda})}(1-V(\bar{\lambda})-\lambda E[V])\right)}{(1-\rho)\left(\lambda E[V]+\rho M\dfrac{qV(\bar{\lambda})}{1-\bar{q}V(\bar{\lambda})}+\dfrac{qV(\bar{\lambda})}{1-\bar{q}V(\bar{\lambda})}Q_M(1)(1-\rho M-\lambda E[V])\right)}. \tag{39}$$

Substituting Eq. (39) and $H(V(\bar{\lambda}))$ into Theorem 3.2 and Theorem 3.3, we obtain the P.G.Fs. of the additional queue length and the additional delay.

If $M=1$, the model turns into a pure limited service Geom/G/1 queueing system with multiple adaptive vacations, if $M\to\infty$, the model turns into a gate service Geom/G/1 queueing system with multiple adaptive vacations. By applying Theorem 3.2 and Theorem 3.3, we can obtain the P.G.Fs. of the additional queue length and the additional delay.

V. Conclusions

In this paper, we have presented the a detailed description of a general limited service Geom/G/1 queueing system with multiple adaptive vacations. We derived the P.G.Fs. of the stationary queue length and the waiting time by using an embedded Markov chain method and a regeneration cycle approach. Furthermore, we obtained the probabilities of the server being at the various states of busy, vacation and idle, respectively. Finally, some special cases were given to verify the above results.

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