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A note on self-fusion of torus knots

Kanji Morimoto

Department of Information Science and Systems Engineering,Konan University
Okamoto 8-9-1, Higashi-Nada, Kobe 658-8501, Japan
e-mail : morimoto@konan-u.ac.jp

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Abstract

For a given torus knot in a standard torus in S^3, we can make another torus knot by performing a self-fushion along a fusion arc in the torus. In this note, we study the relationship between those two knots, and show that those two knots are related by some number theoretic conditions.

1. Definition of tours knots and results

Let $S^1 \times D^2$ be a standard solid torus in S^3, and let $T = S^1 \times S^1$ be the boundary torus. Then a simple closed curve $S^1 \times \{0\}$ in T is called a longitude and denoted by ℓ, and a simple closed curve $\{0\} \times S^1$ in T is called a meridian and denoted by m. Any simple closed curve in T is a simple closed curve in S^3, and hence it is a knot in S^3. Therefore we call a simple closed curve in T which winds p times along ℓ and winds q times along m a torus knot of type (p,q) and denote it by $K(p,q)$. Then ℓ is $K(1,0)$ and m is $K(0,1)$ as indicated in Figure 1.

In general, p or q may not be possitive, but in this note, for convenience, we assume $p \geq 0$ and $q \geq 0$. Since ℓ intersects m once and m intersects ℓ once, we can say that $K(p,q)$ intersects the meridian p times and intersects the longitude q times because $p \geq 0$ and $q \geq 0$. For more detail on torus knots, see [1].

On torus knots, first we have the following :

Proposition 1.1 *Let $K(p,q)$ be a torus knot of type (p,q), then $\gcd(p,q) = 1$. Conversely, for any pair of coprime integers (p,q), there is a torus knot of type (p,q).*

This proposition is the most basic fact on torus knots, and the proof is well known, so we omit it. But we show how to construct a torus knot for a given coprime integeres (p,q). Then, through the construction, the readers will naturally see the necessity and the sufficiency of the condition $\gcd(p,q) = 1$.

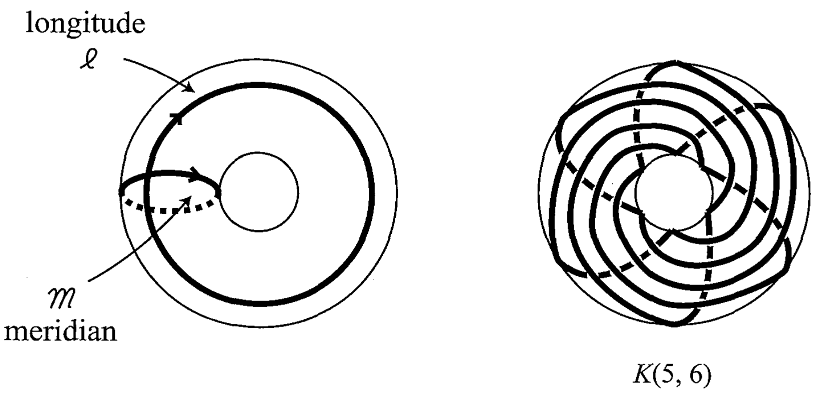


Figure 1

First divide the longitude of the torus qth equally. Then we have q meridians, and on each meridian, mark p points. Then start from the top of the torus, and run clockwise on the torus and connect those p points by shifting once every $\frac{2\pi}{q}$ rotation. Then by the condition $\gcd(p,q)=1$, all $p\cdot q$ points are connected and we have the torus knot $K(p,q)$ at the end of this travel. Figure 2 shows a halfway of $K(5,6)$ and the completion of $K(5,6)$ is illustrated in Figure 1.

Suppose $p>1$. Then since $K(p,q)$ winds more than once along ℓ, there is a subarc of m which interesects $K(p,q)$ in exactly two points in the both ends of the arc. We call this arc a fusion arc. Then by performing a self-fusion of $K(p,q)$ along this arc, we get a new simple closed curve in T as in Figure 3. Then this

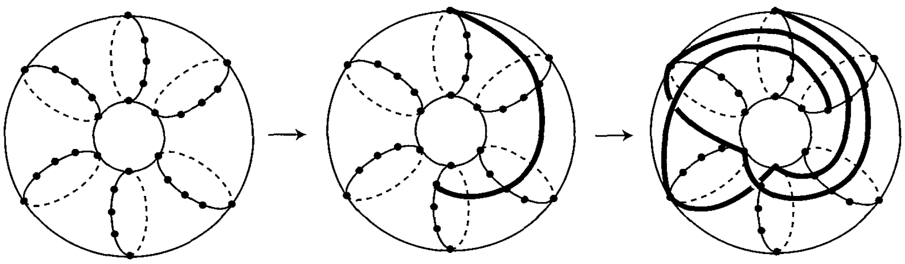


Figure 2

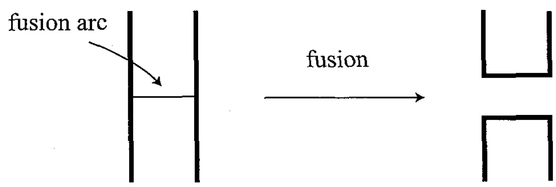


Figure 3

simple closed curve is called a torus knot which is obtained from $K(p,q)$ by a self-fusion (or a fusion simply) along the fusion arc. By this fusion, we get $K(3,4)$ from $K(5,6)$ (Figure 4), and $K(1,1)$ from $K(5,7)$ (Figure 5).

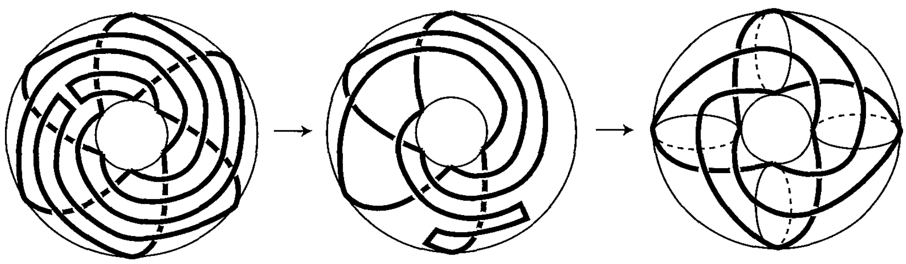
In general, for a given torus knot $K(p,q)$, what kind of torus knot do we get by this fusion ? The purpose of this note is to make it clear, and we get the following. The proof will be given in the next section.

Theorem 1.2 *Let (p,q) be a pair of integers with $p>1$, $q>0$ and $\gcd(p,q)=1$. Suppose $K(a,b)$ is a torus knot obtained from $K(p,q)$ by the self-fusion. Then $\gcd(a,b)=1$ and (a,b) satisfies the following condition $(*)$:*

$$0 \leq a < p-1, \quad pb - qa = \pm 2 \quad \cdots (*)$$

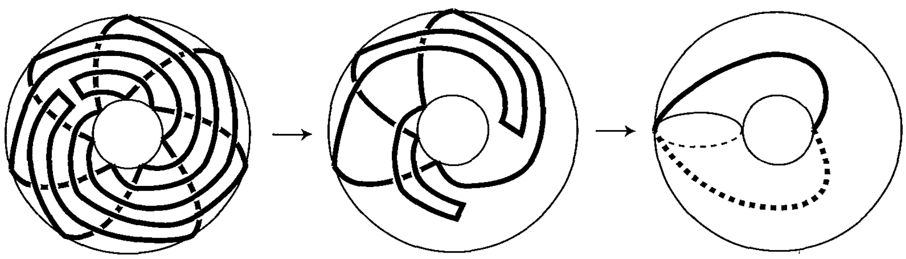
In this case, we have $0 \leq b \leq q$, and by Lemma 2.2, we see that $K(a,b)$ is uniquely determined from $K(p,q)$.

Remark The uniqueness of $K(a,b)$ is shown by geometric argument, but Lemma 2.2 gives an algebraic guarantee of the uniqueness.



K(5,6) fusion and sliding K(3,4)

Figure 4



K(5,7) fusion and sliding K(1,1)

Figure 5

Corollary 1.3 *The necessary and sufficient condition to get a trivial knot from $K(p,q)$ by exactly once fusion is that $p=kq\pm 2$ for some k or $q=kp\pm 2$ for some k.*

Corollary 1.4 (1) *For a given p, the necessary and sufficient condition to get a trivial knot from $K(p,q)$ by maximal times of fusions is that $q=kp\pm 1$ for some k.*

(2) *For a given q, the necessary and sufficient condition to get a trivial knot from $K(p,q)$ by maximal times of fusions is that $p=kq\pm 1$ for some k.*

Examples Put $p=13$, then the sequence to get a trivial knot from $K(p,q)$ by fusions for $q=12,11,\cdots,2$ is the following.

(1) $q=12: K(13,12)\to K(11,10)\to K(9,8)\to K(7,6)\to K(5,4)\to K(3,2)\to K(1,0)$
(2) $q=11: K(13,11)\to K(1,1)$
(3) $q=10: K(13,10)\to K(5,4)\to K(3,2)\to K(1,0)$
(4) $q=9: K(13,9)\to K(7,5)\to K(1,1)$
(5) $q=8: K(13,8)\to K(3,2)\to K(1,0)$
(6) $q=7: K(13,7)\to K(9,5)\to K(5,3)\to K(1,1)$
(7) $q=6: K(13,6)\to K(9,4)\to K(5,2)\to K(1,0)$
(8) $q=5: K(13,5)\to K(3,1)$
(9) $q=4: K(13,4)\to K(7,2)\to K(1,0)$
(10) $q=3: K(13,3)\to K(5,1)$
(11) $q=2: K(13,2)\to K(1,0)$

2. Proof of Theorem 1.2

To prove Theorem 1.2, we prepare three lemmata. In particular, Lemma 2.1 is proved by using Euclidean algorithm.

Lemma 2.1 *Let (p,q) be a pair of integers with $\gcd(p,q)=1$. Then for any integer n, there is a pair of integers (a,b) with $\gcd(a,b)=1$ and $pb-qa=n\cdots$ ①.*

Proof. Let r_1 be the quotient and s_1 the remainder by dividing p by q. Then $p=qr_1+s_1$ and $\gcd(q,s_1)=1$. Suppose, for the pair (q,s_1), we can find (a_1,b_1) with $\gcd(a_1,b_1)=1$ and $qb_1-s_1a_1=n\cdots$ ②. Then, since $s_1=p-qr_1$ and by ②, we have $qb_1-(p-qr_1)a_1=n$ and hence $-pa_1-q(-b_1-r_1a_1)=n$. Moreover, since gcd $(-b_1-r_1a_1,-a_1)=1$, by putting $a=-b_1-r_1a_1$, $b=-a_1$, we have a pair of integers (a,b) which satisfies ①.

Next, to find (a_1,b_1), let r_2 be the quotient and s_2 the remainder by dividing q by s_1. Then $q=s_1r_2+s_2$ and $\gcd(s_1,s_2)=1$. Suppose, for the pair (s_1,s_2), we can find (a_2,b_2) with $\gcd(a_2,b_2)=1$ and $s_1b_2-s_2a_2=n\cdots$ ③. Then, since $s_2=q-s_1r_2$ and by ③, we have $s_1b_2-(q-s_1r_2)a_2=n$ and hence $-qa_2-s_1(-b_2-r_2a_2)=n$.

Moreover, since $\gcd(-b_2-r_2a_2, -a_2)=1$, by putting $a_1=-b_2-r_2a_2$, $b_1=-a_2$, we get a pair of integers (a_1, b_1) which satisfies ②.

As you see above, to get a pair (a, b), it is sufficient to get a pair (a_k, b_k) for some stage. This is called an Euclidean algorithm, and if you continue the procedure $(p, q) \to (q, s_1) \to (s_1, s_2) \to \cdots$, then it is well known that this division sequence ends at the division by $\gcd(p, q)$. Then, since $\gcd(p, q)=1$, we can put $s_{k-1}=1$ and $s_{k-2}=s_{k-1}r_k+s_k$ with $s_k=0$ for some integer k. Then, since $(s_{k-1}, s_k)=(1,0)$, we can put $(a_k, b_k)=(n-1, n)$. Then, by $\gcd(a_k, b_k)=1$ and $s_{k-1}b_k-s_ka_k=1\cdot n-0\cdot(n-1)=n$, we get the pair (a_k, b_k), and by tracing back the above procedures, we get the required pair (a, b). This completes the proof Lemma 2.1. □

Lemma 2.2 *Let (p, q) be the pair of integers with $p>1, q>0$ and $\gcd(p, q)=1$. Then, there uniquely exists a pair of integers (a, b) with $0\leq a<p-1$, $0\leq b\leq q$, $\gcd(a, b)=1$ and $pb-qa=\pm 2$.*

Proof. In Lemma 2.1, put $n=2$, then we see that there is a pair (a, b) with $\gcd(a, b)=1$ and $pb-qa=2$. Suppose $a<0$, then we can take $(-a, -b)$ instead of (a, b). Hence we can assume that $a\geq 0$ and $pb-qa=\pm 2$.

Case 1 : Suppose $a=0$. Since $\gcd(a, b)=1$, $b=\pm 1$ and $pb-qa=pb=\pm 2$. Hence by putting $b=1$, we have the required pair (a, b).

Case 2 : Suppose $0<a<p-1$. If $b<0$, then by $p>1$, $q>0$ we have $pb-qa\leq -2-1=-3$. This contradicts $pb-qa=\pm 2$. Hence $b\geq 0$. If $b>q$, then we have $pb-qa>pq-qa=q(p-a)\geq 2q\geq 2$ and this contradicts $pb-qa=\pm 2$. Therefore $0\leq b\leq \mathrm{q}$, and we have the required pair (a, b).

Case 3 : Suppose $a=p-1$. Put $a_1=p-a=1$, $b_1=q-b$, then $pb_1-qa_1=p(q-b)-q(p-\mathrm{a})=-(pb-qa)=\pm 2$, and $\gcd(a_1, b_1)=1$ by $a_1=1$. Moreover we have $0\leq b_1\leq q$ by the argument in Case 2. If $a_1=1<p-1$, then (a_1, b_1) is the required pair. If $a_1=1=p-1$, then $p=2$ and $pb-qa=2b-q=\pm 2$. Then q is even, and this is a contradiction because of $\gcd(p, q)=1$. Thus $a_1=1<p-1$ and we get the required pair.

Case 4 : Suppose $a>p-1$. Put $a_1=a-kp$, $b_1=b-kq$ for some k, then $pb_1-qa_1=p(b-kq)-q(a-kp)=pb-qa=\pm 2$. Hence by using (a_1, b_1) for some k instead of (a, b), we have $0\leq a_1\leq p-1$. Moreover, if $a_1=p-1$ then this case is done in Case 3, and hence we may assume $0\leq a_1<p-1$.

Case 4-1 : Suppose $\gcd(a_1, b_1)=1$. Then by the arguments in Cases 1, 2, we have $0\leq b_1\leq q$ and (a_1, b_1) is the required pair.

Case 4-2 : Suppose $\gcd(a_1, b_1)\neq 1$. Since $pb_1-qa_1=\pm 2$, we have $\gcd(a_1, b_1)=2$. Put $a_2=p-a_1$ and $b_2=q-b_1$. Then $pb_2-qa_2=p(q-b_1)-q(p-a_1)=-(pb_1-qa_1)=-(pb-qa)=\pm 2$. If $\gcd(a_2, b_2)\neq 1$, then $\gcd(a_2, b_2)=2$. Thus we have $a_1=2a'$, $b_1=2b'$, $a_2=2a''$, $b_2=2b''$, and since $a_2=p-a_1$, $b_2=q-b_1$, we have $2a''=p-2a'$,

$2b'' = q - 2b'$. This means that both of p and q are even numbers, and this contradicts $\gcd(p,q) = 1$. Therefore $\gcd(a_2, b_2) = 1$ and $0 \leq a_2 < p-1$. Then, since $0 \leq b_2 \leq q$ is proved in Cases 1, 2, we see that (a_2, b_2) is the required pair.

In the above arguments, we have shown the existence of the required pair of integers. Next we show the uniqueness, and to do this, suppose there exists another pair of integers (a', b') which satisfies the required conditions. In this case, without loss of generality, it is sufficient to consider the follwoing two cases.

$$\text{Case 1}: \begin{cases} pb - qa = 2 \\ pb' - qa' = 2 \end{cases} \qquad \text{Case 2}: \begin{cases} pb - qa = 2 \\ pb' - qa' = -2 \end{cases}$$

Suppose we are in Case 1. By doing subtraction in both sides, we have $p(b-b') = q(a-a')$. If $a - a' = 0$, then $b - b' = 0$ and we have $(a,b) = (a',b')$. If $a - a' \neq 0$, then we have $\frac{b-b'}{a-a'} = \frac{q}{p}$. Since $\frac{q}{p}$ is an irreducible fraction, we have $a - a' = pk$, $b - b' = qk$ for some k and $|a - a'| \geq$ p. On the other hand, since $0 \leq a < p-1$, $0 \leq a' < p-1$, we have $|a - a'| < p-1$. This contradiction shows that $a - a' = 0$, and we have $(a,b) = (a',b')$.

Suppose we are in Case 2. By doing addition in both sides, we have $p(b+b') = q(a+a')$. If $a + a' = 0$, then $a = a' = b = b' = 0$, a contradicion. Hence $a + a' \neq 0$, $\frac{b+b'}{a+a'} = \frac{q}{p}$ and we have $a + a' = pk$, $b + b' = qk$ for some k. On the other hand, since $0 \leq a < p-1$, $0 \leq a' < p-1$, we have $a + a' < 2p-2$. Then we have $a + a' = p$ and we have $b + b' = q$ similarly.

Suppose p is an odd number. If a is an even number, then b is an odd number by $\gcd(a,b) = 1$. Hence $pb - qa =$ (odd number)$\cdot$(odd number)$- q\cdot$(even number)$=$ (odd number). This contradicts that $pb - qa = \pm 2$. If a is an odd number, then a' is an even number, and we have a contradiction similarly. Next suppose p is an even number. Then q is an odd number by $\gcd(p,q) = 1$, and we have a contradiction similarly. Therefore Case 2 does not occur, and this completes the proof of Lemma 2.2. □

Lemma 2.3 *Let $K(p_1, q_1)$ and $K(p_2, q_2)$ be two torus knots in a standard torus in S^3. Then the minimal intersecting number of $K(p_1, q_1)$ and $K(p_2, q_2)$ is $|p_1 q_2 - q_1 p_2|$, where the minimal intersecting number is the minimal number among all intersecting numbers between two knots in the torus isotopic to $K(p_1, q_1)$ and $K(p_2, q_2)$ respectively.*

Proof. It is well known that there is a self-homeomorphism of T which takes $K(p_1, q_1)$ to ℓ. Hence we denote the self-homeomorphism by $h: T \to T$ and we denote the automorphism on $H_1(T)$ induced from h by $h_*: H_1(T) \to H_1(T)$. Since $H_1(T) = Z + Z$, $K(p_1, q_1)$ and ℓ correspond to (p_1, q_1) and $(1,0)$ in $H_1(T)$ respectively. Then $h_*(p_1, q_1) = (1,0)$ and h_* is represented by a quadratic regular matrix on the integer ring.

Thus we put the matrix $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$, and we have $\begin{bmatrix} r & s \\ t & u \end{bmatrix}\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Since $ru-st=\pm 1$, by changing the h if necessary, we may assume $ru-st=1$. Then by multiplying the inverse matrix from the left side, we have

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}=\begin{bmatrix} u & -s \\ -t & r \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, since $p_1=u$, $q_1=-t$ and by multiplying the first matrix to $[p_2, q_2]$, we have

$$\begin{bmatrix} r & s \\ -q_1 & p_1 \end{bmatrix}\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}=\begin{bmatrix} rp_2+sq_2 \\ -q_1p_2+p_1q_2 \end{bmatrix}$$

This means that $K(p_2,q_2)$ is taken to $K(rp_2+sq_2,\ -q_1p_2+p_1q_2)$ by h. Hence the minimal intersecting number of $K(p_1,q_1)$ and $K(p_2,q_2)$ coincides with the minimal intersecting number of $K(1,0)$ and $K(rp_2+sq_2,-q_1p_2+p_1q_2)$, and this is equal to the absolute value of the second element of $K(\cdot,\cdot)$. Hence the minimal intersecting number of $K(p_1,q_1)$ and $K(p_2,q_2)$ is $|p_1q_2-q_1p_2|$. This completes the proof of Lemma 2.3. □

Proof of Theorem 1.2. Suppose the torus knot $K(a,b)$ is obtained from $K(p,q)$ by the fusion as in Figure 6. Then by sliding $K(a,b)$ away from $K(p,q)$ slightly, we see that $K(a,b)$ and $K(p,q)$ intersects in exactly two points. In addition, since the signature of the two points are the same ones, we see that the minimal intersecting number of the two knots is two. Then by Lemma 2.3, we have $pb-qa=\pm 2$.

Moreover, $K(p,q)$ intersects m in p points, but $K(a,b)$ does not meet m in the two ends of the fusion arc, we see that the minimal intersecting number of $K(a,b)$ and m is at most $p-2$. Hence we have $0\leq a<p-1$. In addition, we see that $0\leq b\leq q$ by the arguments in Lemma 2.2. Thus by Lemma 2.2, we see that the pair (a,b) is uniquely determined by (p,q), and hence $K(a,b)$ is uniquely determined by $K(p,q)$. This completes the proof of Theorem 1.2. □

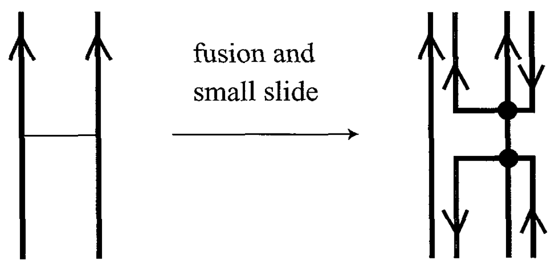


Figure 6

Proof of Corollay 1.3. Since $K(a,b)$ is a trivial knot if and only if $a=1$ or $b=1$ (note that $0 \leq a$, $0 \leq b$), we have $p=aq\pm 2$ or $q=bp\pm 2$ because of $pb-qa=\pm 2$. This completes the proof of Corollary 1.3. □

Proof of Corollary 1.4. First fix an integer p, and suppose $K(a,b)$ is obtained from $K(p,q)$. Then since $a<p-1$, to take the maximal times, we have $a=p-2$. By substituting $a=p-2$ for $pb-qa=\pm 2$, we have $2q=p(q-b)\pm 2$. Then $p(q-b)$ is an even number. If p is an odd number, then $q-b$ is an even number, and we can put $q-b=2k$. Then we have $q=kp\pm 1$. If p is an even number, then q is an odd number by $\gcd(p,q)=1$. Moreover, since $a=p-2$ is an even number, b is an odd number by $\gcd(a,b)=1$. Hence $q-b$ is an even number and we have $q=kp\pm 1$. Thus $K(a,b)=\mathrm{K}(p-2,q-2k)$.

Next, put $a'=p-4$, $b'=q-4k$, then we have $ab'-ba'=(p-2)(q-4k)-(q-2k)(p-4)=2q-2kp=2(q-kp)=\pm 2$. Hence we get $K(p-4,q-4k)$ from $K(p-2, q-2k)$ by the fusion. Then by repeating these procedures, we have the sequence $p \to p-2 \to p-4 \to \cdots$. This means that we have the maximal times of fusions when $q=kp\pm 1$, and completes the proof of (1).

In case (2), we have the same situation. In fact, for a given integer q, we have the sequence $q \to q-2 \to q-4 \to \cdots$, and see that we have the maximal times of fusions when $p=kq\pm 1$. This completes the proof of Corollary 1.4. □

References

[1]. D. Rolfsen, "Knots and Links" AMS Chelsea Publishing (1976, 2003)